Power-Law Energy Splitting Generated By Tunneling Between Non-smooth Tori

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Abstract

We discuss the energy splitting $\Delta\epsilon$ of nearly degenerate eigenstates which classically correspond to symmetrically distributed tori in phase space. We find that the naive expectation that the tunneling-induced splitting vanishes faster than any power of $\hbar$ in semi-classical limit actually relies on certain smoothness assumption of the Hamiltonian. The quantum transition between the semi-classical eigenstates will be greatly enhanced when the corresponding degenerate tori in phase space are connected by line(s) where the Hamiltonian is not smooth. The leading term in semi-classical expansion of $\Delta\epsilon$ is derived under the assumption that the the non-smoothness depends only upon x- or p-coordinate, which shows that $\Delta\epsilon$ decays as $\hbar^{k+1}$ when $\hbar \to 0$ with k being the order of non-smoothness. We conjecture that the non-smoothness-enhanced transition and the resulting power-law decay of $\Delta\epsilon$ are typical in non-smooth systems.

1 Introduction

This paper concerns the splitting of semi-classically degenerate energy levels. The nearly degeneracy (ND) is originated from symmetrically distributed tori in phase space while the quantum tunneling between the tori causes the splitting [1]. A well-known example occurs in the one-dimensional symmetric double-well potential. The eigen energies below the top of the barrier cluster into two-fold ND's with energy differences vanish as

$$\Delta\epsilon \sim \hbar^{\alpha} \mathrm{e}^{-S/\hbar} \tag{1.1}$$

when $\hbar \to 0$. When turn to multi-dimensional cases, M. Wilkinson showed that $\Delta\epsilon$ vanishes normally in the same or, in certain situation, even more singular manner as (1.1)[3]. However, dose it is generic that the energy splitting resulted from quantum tunneling is smaller than any power of $\hbar$ in semi-classical limit? Let us see the following example.

Consider the system on one-dimensional circle defined by any of the four Hamiltonians,

$$H_1 = \frac{p^2}{2} + \cos^2 x, \quad H_2 = \frac{p^2}{2} + |\cos x|, \quad H_3 = |p| + \cos^2 x, \quad H_4 = |p| + |\cos x|. \tag{1.2}$$

In classical mechanics, the above Hamiltonians determine similar phase space portraits, particularly, motion at $H \neq 1$ contains two symmetric closed orbits, which imposes a two-fold ND structure on energy spectrum. In quantum mechanics, according to the same symmetries, the total Hilbert space can be decomposed into the direct sum of four invariant subspaces, $\mathcal{S}_{++} \bigoplus \mathcal{S}_{+-} \bigoplus \mathcal{S}_{-+} \bigoplus \mathcal{S}_{--}$, which are spanned by $\{\cos 2nx\}$,$\{\sin 2nx\}$, $\{\sin(2n+1)x\}$ and $\{\cos(2n+1)x\}$ respectively. Correspondingly, the energy spectrum can be labeled by $\{\epsilon_{\mu\nu,k}\}_{k=0}^{\infty}$ with $\mu, \nu \in \{+, -\}$. Simple analysis show that when $\epsilon < 1$,

$$\epsilon_{++,k} \approx \epsilon_{-+,k} < \epsilon_{--,k} \approx \epsilon_{+-,k},$$

while when $\epsilon > 1$,

$$\epsilon_{+-,k-1} \approx \epsilon_{++,k} < \epsilon_{-+,k} \approx \epsilon_{--,k}$$

(see Appendix I). This arrangement of ND is solely determined by the classical phase space structure, regardless of the smoothness of H. However, distinctive behavior of the strength of ND, which is characterized by a much larger, certainly not exponentially small, energy splitting, can be observed in some cases (Fig. 1).

[1] In this paper, the word "tunneling" refers to quantum transition between states that classically correspond to separate tori in phase space [1].

From the four illustrations in Fig.1, we can see that the "exceptional" ND occurs when and only when the corresponding classically degenerate tori (closed orbits) in phase space are connected by line(s) where the Hamiltonian is not smooth. This fact suggests that tunneling between degenerate tori may be greatly enhanced by passage of non-smoothness. In this paper we shall investigate the energy splitting resulted from this non-smoothness-enhanced tunneling. In the following section, we consider the case where ND is related to time reversal symmetry. We obtain a relation between energy splitting and the non-smoothness of potential. In Sec. 4, we give this relation a geometrical interpretation, which leads to a general treatment of energy splitting in a class of non-smooth systems. This is followed by a general discussion.

2 Power-Law Energy Splitting

In this section we study systems in which ND is related to time reversal symmetry. The problem is more tractable since the projection of torus onto coordinate space contains no singularity (caustic). By perturbation method, we obtain an explicit power-law $\hbar$-dependence of energy splitting.

Consider a mechanic system on one-dimensional circle with Hamiltonian $H = E_k(p) + V(x)$, $V(x + 2\pi) = V(x)$. The kinetic energy $E_k(p)$ satisfies $E_k(-p) = E_k(p)$, and, for simplicity, we assume $E_k(0) = 0$, $E_k(\infty) = \infty$ and $\frac{d}{dp}E_k(p) > 0$ when $p > 0$. A familiar example of such kinetic energy is $\frac{1}{2}p^2$. Due to time reversal symmetry, the two classical orbits O_E^+ and O_E^-, one with $p > 0$ and the other with $p < 0$, at $H(x, p) = E > \max_x V(x)$ yield identical action integral, i.e.,

$$\oint_{O_E^+} p dx = \oint_{O_E^-} p dx = S(E). \tag{2.1}$$

Consequently, the Einstein-Brillouion-Keller (EBK) quantization condition $S(E) = 2n\pi\hbar$ predicts a two-fold degenerate level $E = \epsilon_n$. The two semi-classical eigenfunctions are given by

$$\Psi_n^{\pm}(x) = \frac{1}{\sqrt{T_n \dot{x}_n}} \exp[\pm i s_n(x)/\hbar], \tag{2.2}$$

where $s_n(x) = \int_0^x p_n(x')dx'$, $p_n(x) > 0$ is determined by $E_k(p) + V(x) = \epsilon_n$, $\dot{x}_n = \frac{d}{dp}E_k(p)|_{p=p_n(x)}$ is the classical velocity and the normalization constant $T_n = \int_0^{2\pi} \frac{dx}{\dot{x}_n}$ is the period of corresponding classical orbit[2]. (The suffix "n" of ϵ, Ψ, p, $\dot{x}$, T and s will be hereafter dropped out for simplicity.)

Of cause, in general, the two levels do not exactly coincide. The difference between ϵ and the exact eigen energy is of order $o(\hbar)$ in semi-classical limit ($\hbar \to 0, n \to \infty$ while $n\hbar$ is fixed).

In the case that $V(x)$ is not smooth (infinitely differentiable), we have seen in the last section (H_2 and H_4) that the splitting of energy levels ($\Delta\epsilon$) is not exponentially small. It is therefore possible that a non-vanishing $\Delta\epsilon$ will emerge from the higher order semi-classical corrections. If we are only interested in the leading term in $\Delta\epsilon$, however, variational calculation in the space spanned by Ψ^+ and Ψ^- will give the result. We shall consider a simple case that $V(x)$ is a C^{k-1} function and

$$\bigwedge_x^k V(x) \equiv \lim_{x'\to x+0} \frac{d^k}{dx^k} V(x') - \lim_{x'\to x-0} \frac{d^k}{dx^k} V(x') \tag{2.3}$$

is well-defined, which vanishes on $[0, 2\pi]$ except at discrete points $x_j^*, j = 1, ..., N < \infty$. Then elementary calculations show that the energy splitting is given by (see Appendix II)

$$\Delta\epsilon = \frac{\hbar^{k+1}}{2^k T} |\sum_{j=1}^{N} \frac{\exp(2is(x_j^*)/\hbar)}{p^{k+1}\frac{d}{dp}E_k|_{p=p(x_j^*)}} \bigwedge_x^k V(x_j^*)| + o(\hbar^{k+1}) \equiv \Delta\epsilon^{(0)} + o(\hbar^{k+1}). \tag{2.4}$$

Define a dimensionless measurement of ND by $\eta_n = \frac{2\Delta\epsilon_n}{\epsilon_{n+1}-\epsilon_{n-1}}$. Noticing that the average level distance predicted by EBK quantization condition is $2\pi\hbar\frac{dE}{dS} = 2\pi\hbar/T$ and according to Eq. (2.4),

$$\eta = \frac{\hbar^k}{2^{k+1}\pi} |\sum_{j=1}^{N} \frac{\exp(2is(x_j^*)/\hbar)}{p^{k+1}\frac{d}{dp}E_k|_{p=p(x_j^*)}} \bigwedge_x^k V(x_j^*)| + o(\hbar^k) \equiv \eta^{(0)} + o(\hbar^k). \tag{2.5}$$

Example 2.1 $H = \frac{1}{2}p^2 + V^{(k)}(x)$, where $V^{(1)}(x) = \max\{\cos x, 0\}$ and $V^{(k)}(x) = [V^{(1)}(x)]^k$, $k = 2, 3, ...$

According to Eq. (2.5), when $\epsilon > 1$,

$$\eta^{(0)} = \frac{k!\hbar^k}{2^k\pi(2\epsilon)^{\frac{k}{2}+1}} |\sin[\frac{(2\epsilon)^{\frac{1}{2}}\pi}{\hbar} + \frac{k\pi}{2}]|.$$

The comparison of η and $\eta^{(0)}$ is shown in Fig.2.

Example 2.2 $H = |p| + V^{(k)}(x)$.

When $\epsilon > 1$, the semi-classical level is given by $\epsilon_n = n\hbar + \alpha_k$ and according to Eq. (2.4)

$$\Delta\epsilon^{(0)} = \frac{k!\hbar^{k+1}}{2^k\pi\epsilon^{k+1}} |\sin(\frac{\alpha_k\pi}{\hbar} + \frac{k\pi}{2})|$$

where

$$\alpha_k \equiv \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^k x dx = \frac{\Gamma(\frac{k+1}{2})}{2\Gamma(\frac{1}{2})\Gamma(\frac{k}{2}+1)}.$$

The comparison of $\Delta\epsilon$ and $\Delta\epsilon^{(0)}$ is shown in Fig.3.

3 Sum Over Transition Paths

In this section we first give Eq. (2.4) a geometrical interpretation. We find the quantum transition between the semi-classical eigenstates can be classically described by the leaking of phase space points from one torus to another via passage of non-smoothness. This picture will facilitate the generalization of Eq. (2.4).

The splitting of nearly degenerate eigen energy is closely related to the transition probability between the corresponding quasi eigenstates. Let $\mathcal{A}$ be the $\Psi^+ \to \Psi^-$ transition amplitude accumulated in one classical period. Simple calculation shows that $|\mathcal{A}| \approx \sin(\Delta\epsilon T/2\hbar) \approx \Delta\epsilon T/2\hbar$. According to Eq. (2.4), up to an arbitrary phase,

$$\mathcal{A} \approx \mathcal{A}^{(0)} = \sum_{j=1}^{N} \frac{\hbar^k \exp(2is(x_j^*)/\hbar)}{(2p_j^*)^{k+1}\dot{x}_j^*} \bigwedge_x^k V(x_j^*) = \sum_{j=1}^{N} \mathcal{A}_j^{(0)}, \tag{3.1}$$

where $p_j^* \equiv p(x_j^*)$ and $\dot{x}_j^* \equiv \frac{d}{dp}E_k|_{p=p(x_j^*)}$. Eq. (3.1) suggests that the leading contribution of the non-smoothness at $x = x_j^*$ can be identified as

$$\mathcal{A}_j^{(0)} = \frac{\hbar^k \exp(2is(x_j^*)/\hbar)}{(2p_j^*)^{k+1}\dot{x}_j^*} \bigwedge_x^k V(x_j^*) \equiv d_j \exp(i\phi_j). \tag{3.2}$$

Heuristically, Ψ^+ (Ψ^-) is the quantum counterpart of the time-invariant distribution of phase space points established on torus O_ϵ^+ (O_ϵ^-). O_ϵ^+ and O_ϵ^- are connected by the straight line $x = x_j^*$ where H is not smooth. We shall call the vector on $x = x_j^*$ that starts from O_ϵ^+ and ends at O_ϵ^- a *transition path* and denote it by γ_j (Fig.4). Accordingly, we can say that $\Psi^+ \to \Psi^-$ is dominated by tunneling along transition path(s). It is reasonable to expect that $\mathcal{A}_j^{(0)}$ is determined by quantities of γ_j. Besides a constant, the amplitude d_j consists of three ingredients. $\bigwedge_x^k V(x_j^*)$ can be regarded as the intensity of non-smoothness at γ_j. $\frac{1}{(2p_j^*)^{k+1}}$ describes the power-law decay of d_j with the increase of path length $2p_j^* = \hbar\frac{\partial\phi_j}{\partial x_j^*}$. $\frac{1}{\dot{x}_j^*}$, which comes from the product of amplitude of semi-classical wave functions, can be regarded as the relative populate probability of γ_j. In contrast to d_j, the phase ϕ_j is not determined by local quantities of γ_j. Since only relative phase is of physical importance, i.e., gives rise to interference effect, we find

$$\phi_j - \phi_k = \frac{2}{\hbar}(s(x_j^*) - s(x_k^*)) = \frac{1}{\hbar}\oint_{\gamma_{jk}} p dx, \tag{3.3}$$

where γ_{jk} is a closed path consists of γ_j, -γ_k (γ_k with opposite direction) and the segments O_ϵ^+ and O_ϵ^- (real path) attached at their ends (see Fig.4). If γ_{jk} is contractible then ϕ_{jk} is just the phase space area (in the unit of $\hbar$) enclosed by this closed path.

Behind the simple form of Eq. (3.2) there are two non-generic facts resulted from the assumption that $\frac{d}{dp}E_k(p) > 0$ when $p > 0$: The starting and end points of γ_j are symmetric with respect to $p = 0$ and the projection of O_ϵ^+ or O_ϵ^- onto coordinate space contains no singularity (caustic). Now we ignore this assumption and require only $E_k(-p) = E_k(p)$ to guarantee time reversal symmetry. Let $A_j = (x_j^*, p_j) \in O_\epsilon^+$ and $A'_j = (x_j^*, p'_j) \in O_\epsilon^-$ be the starting and end points of γ_j. By adopting the general semi-classical eigenfunctions corresponding to tori O_ϵ^+ and O_ϵ^- [2], similar calculations as that performed in Appendix II show that Eq. (3.2-3) should be modified as

$$\mathcal{A}_j^{(0)} = \frac{\hbar^k \exp(i\phi_j)}{(p_j - p'_j)^{k+1}\sqrt{|\dot{x}(A_j)\dot{x}(A'_j)|}} \bigwedge_x^k V(x_j^*) \tag{3.4}$$

and

$$\phi_j - \phi_k = \frac{1}{\hbar}\oint_{\gamma_{jk}} p dx - M_{jk}\pi/2, \tag{3.5}$$

where M_{jk} is the sum of Maslov indices of the segments of real paths on γ_{jk}. Having the contribution of each transition path, we need only to sum over all these paths to obtain the transition amplitude $\mathcal{A}^{(0)}$ or energy splitting $\Delta\epsilon^{(0)}$ or $\eta^{(0)}$.

Example 3.1 $H = (p^2 - 1)^2 + V(x)$, where $V(x) = 1 - (\frac{x}{\pi})^2, |x| \le \pi$.
When $\epsilon < 1$, the Maslov index of O_ϵ^+ (or O_ϵ^-), which encircles point $(\pi, 1)$ (or $(\pi, -1)$), is 2 and EBK quantization condition reads $S(\epsilon_n) = 2(n + \frac{1}{2})\pi\hbar$ (see inset of Fig. 5). Straight line $x = \pi$ intersects O_ϵ^+ (or O_ϵ^-) at points $A_{1,2}$ (or $A'_{1,2}$) where $p = (1 \pm \epsilon^{\frac{1}{2}})^{\frac{1}{2}}$ (or $-(1 \pm \epsilon^{\frac{1}{2}})^{\frac{1}{2}}$), which produce four transition paths, i.e., γ_1 ($A_1 \to A'_1$), γ_2 ($A_2 \to A'_2$), γ_3 ($A_1 \to A'_2$) and γ_4 ($A_2 \to A'_1$). Moreover, EBK quantization condition implies $\phi_j - \phi_1 = 0, n\pi, n\pi \pmod{2\pi}$ for $j = 2, 3, 4$ respectively. According to Eq. (3.4),

$$\mathcal{A}^{(0)} = \frac{\hbar}{4\pi\epsilon^{\frac{1}{2}}}\left[\frac{1}{(1+\epsilon^{\frac{1}{2}})^{\frac{3}{2}}} + \frac{1}{(1-\epsilon^{\frac{1}{2}})^{\frac{3}{2}}} + (-1)^n \frac{4}{(1+(1-\epsilon)^{\frac{1}{2}})(1-\epsilon)^{\frac{1}{4}}}\right].$$

When $\epsilon > 1$, only γ_1 survives and according to Eq.(3.4),

$$\mathcal{A}^{(0)} = \frac{\hbar}{4\pi\epsilon^{\frac{1}{2}}(1+\epsilon^{\frac{1}{2}})^{\frac{3}{2}}}.$$

Numerical results show that $\eta^{(0)} = |\mathcal{A}^{(0)}|/\pi$ is a good approximation of η when $\hbar$ is sufficiently small and ϵ is not too close to 1, the energy of separatrix(Fig. 5).

The same treatment can be applied to non-smooth systems where ND is originated from spatial symmetries. By substituting $(p, -x) \to (x, p)$, relations (3.4-5) can be directly transformed to systems where the non-smoothness that results transition path depends only upon

p-coordinate. Specifically, suppose there is a transition path γ_j on straight line $p = p_j^*$ with starting and end points being respectively $A_j = (x_j, p_j^*)$ and $A_j' = (x_j', p_j^*)$, its leading contribution should be

$$\mathcal{A}_j^{(0)} = \frac{\hbar^k \exp(i\phi_j)}{(x_j' - x_j)^{k+1}\sqrt{|\dot{p}(A_j)\dot{p}(A_j')|}} \bigwedge_p^k H(x, p_j^*). \tag{3.6}$$

The phase difference is also given by Eq. (3.5), whereas the Maslov index should count the singularity of the projection of torus onto momentum space [2]. Despite this similarity, distinctive behavior may occur due to non-trivial topology of the configuration space. We shall demonstrate it by some examples.

Suppose the configuration space is a circle, i.e., (x, p) and $(x+2\pi, p)$ describe the same point. In this case, winding in coordinate space by an arbitrary additional loops will produce a family of transition paths from A_j to A_j^*. The total contribution of all these paths can be obtained by replacing $\frac{1}{(x_j' - x_j)^{k+1}}$ in Eq. (3.6) with

$$\sum_{q=-\infty}^{\infty} \frac{\exp(i2q\pi p_j^*/\hbar)}{(x_j' - x_j + 2q\pi)^{k+1}} \equiv W_{k+1}(x_j' - x_j, p_j^*/\hbar). \tag{3.7}$$

W satisfies periodic condition $W_k(x, y+1) = e^{i2\pi y} W_k(x + 2\pi, y) = W_k(x, y)$. When $y \in [0, 1]$,

$$\begin{aligned} W_2(x, y) &= \frac{1}{4\sin^2\frac{x}{2}}[1 + y(e^{ix} - 1)]e^{-ixy}, \\ W_3(x, y) &= \frac{1}{8\sin^3\frac{x}{2}}[\cos\frac{x}{2} + i2y\sin\frac{x}{2} - 2y^2\sin^2\frac{x}{2}e^{ix/2}]e^{-ixy} \end{aligned} \tag{3.8}$$

and so on. We note that $\mathcal{A}^{(0)}$ is in general not invariant under translation $(x, p) \to (x, p + \delta_p)$ when δ_p is not an integer multiple of $\hbar$, which is however always a symmetric transformation in classical mechanics. This difference reflects the discreteness of quantum momentum space.

Example 3.2 $H = |p - p_c| + \cos^2 x$.

The symmetric double-well potential causes ND at $\epsilon < 1$. According to Eq. (3.6-7), the total contribution of transition paths (on $p = p_c$) is given by

$$\mathcal{A}^{(0)} = \frac{\hbar}{\epsilon^{\frac{1}{2}}(1-\epsilon)^{\frac{1}{2}}}[W_2(2x_c, \frac{p_c}{\hbar}) + W_2(2\pi - 2x_c, \frac{p_c}{\hbar}) + (-1)^n 2W_2(\pi, \frac{p_c}{\hbar})],$$

where $x_c = \cos^{-1}\epsilon^{\frac{1}{2}}$. When $p_c = 0$, $\mathcal{A}^{(0)} = \frac{\hbar}{2\epsilon^{\frac{1}{2}}(1-\epsilon)^{\frac{1}{2}}}[\frac{1}{1-\epsilon} + (-1)^n]$. When $p_c = \frac{\hbar}{2}$, $\mathcal{A}^{(0)} = 0$. In fact, $\Delta\epsilon \equiv 0$ in this case because restricting H to the invariant subspace $S_{++} \bigoplus S_{+-}$ or $S_{-+} \bigoplus S_{--}$ yields identical matrix.

[2] We use $\omega_1 = pdx$ instead of $\omega_1' = -xdp$ based on two facts. Firstly, $\oint_{\gamma_{jk}} \omega_1 = \oint_{\gamma_{jk}} \omega_1'$ when γ_{jk} is contractible. Secondly, if the coordinate space has non-trivial topology, ω_1 is well-defined while ω_1' is not. We find that this choice is justified by numerical results.

Consider a spin system defined in classical and quantum mechanics by respectively $\{J_j, J_k\} = \varepsilon_{jks} J_s$ and $[J_j, J_k] = i\hbar\varepsilon_{jks} J_s,\ j,k = 1,2,3$. When $J^2 = J_1^2 + J_1^2 + J_3^2$ is fixed, the classical mechanics is confined within a sphere $\mathcal{S}_J$. Restricting the su(2) Poisson structure to $\mathcal{S}_J$ yields a sympletic two form $\omega_2 = J\sin\theta d\phi \wedge d\theta$, where (θ, ϕ) is the conventional sphere coordinate. In quantum mechanics, $J^2 = j(j+1)\hbar^2$, $j = \frac{1}{2}, 1, \frac{3}{2}$.... An eigenspace of J^2 is associated with a classical sphere $\mathcal{S}_J$, in which we shall assume $J = (j+\frac{1}{2})\hbar$ so that its phase area (integral of ω_2 on $\mathcal{S}_J$) in unit $2\pi\hbar$ is $2j+1$, which corresponds to the dimension of the eigenspace. In our treatment of non-smooth systems, a prerequisite is that the phase space is the direct product of coordinate and momentum spaces. To meet this requirement, we write $(J\cos\theta + p_0, \phi) = (p, x)$, in which $\omega_2 = dp \wedge dx$, and regard (x, p) as the nature coordinate of the phase space of a mechanic system on circle. Moreover, to ensure the right spectrum of $J_3 = p - p_0$, we choose $p_0 = 0$ (or $\frac{1}{2}\hbar$) in the case of j is an integer (or half integer). By this transformation in classical mechanics, we can treat the non-smoothness-enhanced tunneling in some spin systems.

Example 3.3

$$H(J_1, J_2, J_3) = \begin{cases} J_1^2 - J_2^2 + J_3^2 & J_3 \geq 0, \\ J_1^2 - J_2^2 & J_3 < 0. \end{cases}$$

The corresponding classical system on circle is

$$H(x, p) = \begin{cases} [J^2 - (p-p_0)^2]\cos 2x + (p-p_0)^2 & p \geq p_0, \\ [J^2 - (p-p_0)^2]\cos 2x & p < p_0. \end{cases}$$

From phase space portrait we know that energy levels in $(-J^2, 0)$ consist of 2-fold ND and according to Eq. (3.6-7),

$$\mathcal{A}^{(0)} = \frac{\hbar^2}{J^2 \sin 2x_c}[W_3(2\pi - 2x_c, \frac{p_0}{\hbar}) + W_3(2x_c, \frac{p_0}{\hbar})e^{2i(\phi - \frac{\pi}{2})} + 2W_3(\pi, \frac{p_0}{\hbar})e^{i(\phi - \frac{\pi}{2})}].$$

where $x_c = \frac{1}{2}\cos^{-1}\xi$ with $\xi \equiv \epsilon/J^2$ and $\phi = \pi J(1 - \sin x_c)/\hbar = \pi(j+\frac{1}{2})[1 - (\frac{1-\xi}{2})^{\frac{1}{2}}]$. When j is an integer,

$$|\mathcal{A}^{(0)}| = \frac{|\cos\phi|}{2(j+\frac{1}{2})^2(1-\xi)^2}$$

while when j is a half integer,

$$|\mathcal{A}^{(0)}| = \frac{1}{4(j+\frac{1}{2})^2(1-\xi^2)^{\frac{1}{2}}}|\frac{3+\xi}{\sqrt{2}(1-\xi)^{\frac{3}{2}}}\sin\phi + \frac{1}{2}|$$

(In this case, $\frac{1}{j+\frac{1}{2}}$ can be regarded as an efficient $\hbar$.) These relations give a good description of the energy splitting when $j \gg 1$ (Fig. 6).

4 Discussion

We have derived, under some restrictions, the energy splitting generated by non-smoothness-enhanced tunneling. It is reasonable to expect that this power-law $\hbar$-dependence of $\Delta\epsilon$ is typical in general non-smooth systems, e.g., the system may be multi-dimensional and the order of non-smoothness may be any positive real number, the non-smoothness may depend on both coordinate and momentum variables or the phase space may be a general sympletic manifold where a global distinguish of coordinate and momentum is impossible. Another interesting manifestation of non-smoothness is the power-law localization of eigenstates, which is proved to be characteristic in one-dimensional non-smooth system[4] and also observed in periodically kicked system[5]. However, it should pointed out that the power-law localization of eigenstates (or decay of off-diagonal elements of Hamiltonian matrix) is not a representation-independent description. For instance, when the transition path is perpendicular to $p-$direction, the eigenstates might be highly localized in p-representation (such as Example 3.2).

The essential difference between the non-smoothness-enhanced and the conventional tunneling lie in the decay behaviors of $\Delta\epsilon$. On the other hand, we know that a non-smooth Hamiltonian can be infinitely close to a smooth one. Our conclusion dose not contradict the principle of continuity because there are two separate limit processes are concerned, one is the semi-classical limit $\hbar \to 0$ and the other is the approaching of a give non-smooth system by a series of smooth systems. The relation between the two processes can be clarified in a picture of perturbation. Let us suppose ND is related to time reversal symmetry. The off-diagonal elements of $H = E_k(p) + V(x)$ in momentum representation read $H_{ij} =< i|V|j >\equiv V_{j-i}$. V_m describes the hopping amplitude in momentum space and, when $V(x)$ is not smooth, it vanishes as $|m|^{-(k+1)}$ when $m \to \infty$ (see Eq. (A. 2.4)). If $V(x)$ is treated as a perturbation then the first order contribution reads $\Delta\epsilon = |V_{\frac{\Delta p}{\hbar}}| \sim \hbar^{k+1}$, where Δp is the momentum distance between the unperturbed states. When $V(x)$ is smooth, V_m beyond a bandwidth k_c, which diverges if $V(x)$ is arbitrarily close to a non-smooth function, will approach 0 faster than any power of $|m|^{-1}$. The decay V_m in this case dose not necessarily directly determine the the decay of $\Delta\epsilon$. For simplicity, we assume $V_m = 0$ when $|m| > k_c$. If $k_c\hbar > \Delta p$, the first order perturbation will give the main part of $\Delta\epsilon$, which is essentially the same as that in non-smooth systems. However, this similarity will be broken down by decreasing $\hbar$ to less than $\frac{\Delta p}{k_c}$, when the higher order contributions, i.e., transitions via some intermediate states, must be included. For a given smooth system, when $\hbar \to 0$, the order of perturbation that gives the leading term of $\Delta\epsilon$ will be arbitrarily high and, consequently, $\Delta\epsilon$ will decay faster than any power $\hbar$.

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References

[1] M. C. Gutzwiller, Chaos in Classical and Quantum Mechanics (Springer, New York, 1990).

[2] A.M. Ozorio de Almeida, Hamiltonian system: Chaos and Quantization (Cambridge Uiversity Press, 1988).

[3] M. Wilkinson, Physica 21D 341 (1986); M. Wilkinson and J.H. Hannay, Physica 27D 201 (1987).

[4] A. M. Mirlin, Y. V. Fyodorov, F. M. Dittes, J. Quezada and T. H. Seligman, Phys. Rev. E54, 3221 (1996); Y. V. Fyodorov and A. M. Mirlin Int. J. Mod. Phys. B8 3795 (1994).

[5] B. Hu, B. Li, J. Liu and Y. Gu. Phys. Rev. Lett. 82 4224 (1999).

Figure Captions

Fig.1 Splitting of nearly degenerate energy levels at $\hbar = 0.02$. (a)-(d) for $H = H_1$, H_2, H_3 and H_4 respectively. Open circles, solid lines and dotted lines represent $\Delta\epsilon$ (numerical results), semi-classical energy splitting (by perturbation method) and the spacing of semi-classical levels respectively. The insets show the degenerated tori (solid line) in phase-space where H is not smooth on the dotted line(s).

Fig.2 Scaled energy splitting η (open circles) and $\eta^{(0)}$ (connected solid dots) in Example 2.1 at $k = 1$ to 4 and $\hbar = 0.05$.

Fig.3 Energy splitting $\Delta\epsilon$ (open circles) and $\Delta\epsilon^{(0)}$ (solid lines) in Example 2.2 at $k = 1$ to 4 and $\hbar = 0.04$.

Fig.4 Schematic figure show transition paths γ_1 ($A \to A'$), γ_2 ($B \to B'$) and closed path γ_{21} ($B \to B' \to A' \to A' \to B'$). Semi-classical quantum condition guarantees that $\phi_2 - \phi_1$ (mod 2π) is independent of the choice of real paths $B' \to A'$ on O_ϵ^- and $A \to B$ on O_ϵ^+.

Fig.5 η (open circles) and $\eta^{(0)}$ (solid lines) in Example 3.1 at $\hbar = 0.02$. The inset shows three types of tori in phase space. The tori encircling point $(0, 0)$ produce a semi-classically non-degenerate component of energy spectrum at $1 < \epsilon \leq 2$, which has been excluded according to semi-classical criterion that the expectation value of p^2 at the corresponding eigenstates is less than unity.

Fig.6 η (open circles) and $\eta^{(0)}$ (connected dots) in Example 3.3 at (a) $j = 100$ and (b)$j = 99\frac{1}{2}$.

Appendix I: Phase Space Structure and Nearly Degeneracy

The four Hamiltonians in (1.2) admit identical symmetry group, which can be generated from T_1, T_2 defined by

$$T_1: \quad (x,p) \to (x+\pi, p) \qquad T_2: \quad (x,p) \to (\pi - x, -p). \tag{A.1.1}$$

It can be easily verify that $T_1T_2 = T_2T_1$ and $T_1^2 = T_2^2 = I$. Moreover, the four invariant subspaces are eigenspaces of T_1 and T_2 according to

$$T_1 S_{+\mu} = S_{+\mu}, \quad T_1 S_{-\mu} = -S_{-\mu}, \quad T_2 S_{\mu+} = S_{\mu+}, \quad \text{and} \quad T_2 S_{\mu-} = -S_{\mu-}, \tag{A.1.2}$$

$\mu \in \{+,-\}$.

For oscillatory states ($E < 1$), the semi-classical energy level $E = \epsilon_n$ is determined by $S(\epsilon_n) = 2(n+\frac{1}{2})\pi\hbar$. The two quasi eigenstates Ψ_n^L and Ψ_n^R, $\Psi_n^R(x) = \Psi_n^L(x+\pi)$, are localized within the well centered at $x = \pi/2$ and $x = 3\pi/2$ respectively. T_1, T_2 in the space spanned by Ψ_n^L and Ψ_n^R are represented by matrices

$$T_1^{(n)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad T_2^{(n)} = (-1)^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{A.1.3}$$

Therefore, the correct symmetries of the two exact energy levels close to ϵ_n are $(T_1, T_2) = (1, (-1)^n)$ and $(-1, (-1)^n)$ respectively, which implies that $\epsilon_{++,k}, \epsilon_{-+,k} \approx \epsilon_{2k}$ and $\epsilon_{+-,k}, \epsilon_{--,k} \approx \epsilon_{2k+1}$, $k = 0, 1....$

For rotational states ($E > 1$), the semi-classical energy level $E = \epsilon_n$ is determined by $S(\epsilon_n) = 2n\pi\hbar$. The two quasi eigenstates are localized in momentum space with $p > 0$ and $p < 0$ respectively. As to the classification of energy levels, it helpful to consider the parameterized Hamiltonian $H(p, x; \lambda) = E_k(p) + \lambda V(x)$. When $\lambda = 0$, the degeneracy is exact, i.e., $\epsilon_{++,k} = \epsilon_{+-,k-1} = E_k(2k\hbar)$ and $\epsilon_{-+,k} = \epsilon_{--,k} = E_k((2k+1)\hbar)$. Because both the semi-classical quantization condition for rotational states and the exact energy levels vary continuously with λ, we have $\epsilon_{++,k}, \epsilon_{+-,k-1} \approx \epsilon_{2k}$ and $\epsilon_{-+,k+1}, \epsilon_{--,k} \approx \epsilon_{2k+1}$ when $\lambda = 1$.

Appendix II: Semi-classical Calculation of Energy Splitting

We first consider the conventional Hamiltonian $H = \frac{1}{2}p^2 + V(x)$. Direct calculation show that

$$(-\frac{\hbar^2}{2}\frac{d^2}{dx^2} + V(x))\Psi^{\pm}(x) = (\epsilon + Q(x))\Psi^{\pm}(x), \tag{A.2.1}$$

with $Q = -\frac{\hbar^2}{2}p^{1/2}(p^{-1/2})''$, where the prime denotes derivation with respect to x at fixed ϵ. Because $< \Psi_+|\Psi_+ >=< \Psi_-|\Psi_- >= 1$ and $< \Psi_-|\Psi_+ > \sim 0$, the energy splitting calculated in the space spanned by Ψ^+ and Ψ^- is given by

$$\Delta\epsilon = 2| < \Psi^-|Q|\Psi^+ > | = \frac{\hbar^2}{2T}|\int_0^{2\pi}[\frac{V''}{p^3} + \frac{5(V')^2}{2p^5}]\exp(i2s(x)/\hbar)dx|. \qquad (A.2.2)$$

Before evaluating $\Delta\epsilon$ according to Eq. (A. 2.2), it is helpful to recall an useful mathematical result on asymptotic behavior of the Fourier coefficient of a non-smooth function. Let $f(x)$ be a sufficiently regular 2π-periodic function on R. How its Fourier coefficients, defined by

$$\widehat{f}(n) = \int_0^{2\pi} f(x)\exp(inx)dx, \qquad n \in Z, \qquad (A.2.3)$$

decays when $n \to \pm\infty$ is basically determined by the analytic property of $f(x)$. If it is smooth, then $\widehat{f}(n)$ for large n will approach zero faster than any power of $|n|^{-1}$, i.e., $\lim_{|n|\to\infty}\widehat{f}(n)|n|^{\alpha} = 0$ for arbitrary $\alpha > 0$. On the other hand, if $f(x)$ is not smooth, the decay of $\widehat{f}(n)$ may follow a power law. In the simple case when $f(x)$ is the union of N smooth segments on intervals $[x_i^*, x_{i+1}^*]$, $x_1 < x_2 ... < x_{N+1} = x_1 + 2\pi$, $\widehat{f}(n)$ can be expressed by asymptotic series

$$\widehat{f}(n) = \sum_{l=0}^{\infty}\frac{i^{l+1}}{n^{l+1}}\sum_{j=1}^{N}\exp(inx_j^*)\bigwedge_x^l f(x_j^*). \qquad (A.2.4)$$

Let $s(x) = n\hbar\theta(x)$, we rewrite Eq. (A. 2.2) as

$$\Delta\epsilon = \frac{\hbar^2}{2T}|\int_0^{2\pi}[\frac{V''}{p^3} + \frac{5(V')^2}{2p^5}]\frac{n\hbar}{p}\exp(i2n\theta)d\theta|. \qquad (A.2.5)$$

Noticing the integrand apart from $\exp(i2n\theta)$ is unchanged in semi-classical limit, according to Eq. (A. 2.4), we have

$$\Delta\epsilon = \frac{\hbar^{k+1}}{2^k T}|\sum_{j=1}^{N}\frac{\exp[2is(x_j^*)/\hbar]}{p^{k+2}(x_j^*)}\bigwedge_x^k V(x_j^*)| + o(\hbar^{k+1}). \qquad (A.2.6)$$

Then we consider Hamiltonian $H = E_k(p) + V(x)$. In order to evaluate energy splitting according to $\Delta\epsilon = 2| < \Psi^-|H - \epsilon|\Psi^+ > |$, it is instructive to go into some details about the momentum representation of $\Psi^{\pm}$. Write

$$\Psi^{\pm} = \sum_{r=-\infty}^{\infty}\phi_r^{\pm}|r >, \qquad < x|r >= \frac{1}{\sqrt{2\pi}}\exp(irx), \qquad (A.2.7)$$

with

$$\phi_r^{\pm} = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\Psi^{\pm}(x)\exp(-irx)dx. \qquad (A.2.8)$$

Since $\phi^-_{-r} = \phi^{+*}_r$, we shall focus on Ψ^+. The semi-classical limit of Eq. (A. 2.8) should be calculated in two separate cases. In the classically permissible region (CPR), where $p(x) - r\hbar = 0$ is satisfied by some $x \in [0, 2\pi)$, stationary phase approximation can be adopted, which results

$$\phi^+_r \approx \sqrt{\frac{\hbar}{T}} \sum_m \frac{1}{\sqrt{|V'(x_m)|}} \exp[i(s(x_m)/\hbar - rx_m - \sigma_m \pi/2)], \qquad (A.2.9)$$

where$\{x_m\}$ are solutions of $p(x) - r\hbar = 0$ and $\sigma_m = \text{sign}(V''(x_m))$. When $r\hbar$ is beyond CPR, by using expansion (A. 2.4), we find

$$\phi^+_r \approx \frac{(i\hbar)^{k+1}}{\sqrt{2\pi T}} \sum_{j=1}^{N} \frac{\exp[i(s(x^*_j)/\hbar - rx^*_j)]}{(p - r\hbar)^k \frac{d}{dp} E_k} \frac{d}{dp} [\frac{-1}{(p - r\hbar)\sqrt{\frac{d}{dp} E_k}}]|_{p=p(x^*_j)} \bigwedge^k_x V(x^*_j). \qquad (A.2.10)$$

From Eq. (A. 2.9-10) we conclude that Ψ^+ consists of the main part distributed within CPR and two power-law-like long tails beyond CPR. (As the non-smoothness of eigenfunction is resulted via eigen equation from $V(x)$, this picture is also true for exact eigenfunction.) Furthermore, if the semi-classical momentum representation of $V\Psi^+$ is calculated in the similar procedure, one can find that the main part of Ψ^+ within CPR but its long tails approximately satisfies eigen equation $(E_k(p) + V(x))\Psi = \epsilon\Psi$, i.e.,

$$\sum_{m=-\infty}^{\infty} (E_k(r\hbar)\delta_{m,0} + V_m)\phi^+_{r+m} \approx \epsilon\phi^+_r, \qquad (A.2.11)$$

when $r\hbar \in$ CPR, where

$$V_m = < 0|V|m > \approx \frac{i^{k+1}}{2m^{k+1}\pi} \sum_{j=1}^{N} \exp(imx^*_j) \bigwedge^k_x V(x^*_j) \quad (|m| \to \infty). \qquad (A.2.12)$$

Based on the above discussion, we know that

$$\begin{aligned} < \Psi^-|E_k(p)|\Psi^+ > &= \sum_{r=-\infty}^{\infty} \phi^+_{-r} E_k(r\hbar)\phi^+_r \approx \sum_{|r\hbar| \in \text{CPR}} \phi^+_{-r} E_k(r\hbar)\phi^+_r \\ &\approx (\sum_{\nu\hbar \in \text{CPR}} \sum_{\mu=-\infty}^{\infty} + \sum_{-\mu\hbar \in \text{CPR}} \sum_{\nu=-\infty}^{\infty}) \phi^+_{-\nu}(\epsilon\delta_{\nu,\mu} - V_{\mu-\nu})\phi^+_\mu \end{aligned} \qquad (A.2.13)$$

Compare the last expression with

$$< \Psi^-|\epsilon - V|\Psi^+ > = \sum_{\mu,\nu=-\infty}^{\infty} \phi^+_{-\nu}(\epsilon\delta_{\nu,\mu} - V_{\mu-\nu})\phi^+_\mu. \qquad (A.2.14)$$

The main contribution of Eq. (A. 2.14) consists of three parts which come from regions, $(1)\mu\hbar, \nu\hbar \in$ CPR, $(2)-\mu\hbar, -\nu\hbar \in$ CPR and $(3)\mu\hbar, -\nu\hbar \in$ CPR respectively. Eq. (A. 2.13)

contains only the former two parts while we can screen the last contribution by making a high frequency cut off of V(x), i.e., replacing it by

$$V^{(0)}(x) = \sum_{|m|\leq k_c} V_m \exp(-imx), \tag{A.2.15}$$

where k_c is a large but fixed integer so that $V^{(1)}(x) = V(x) - V^{(0)}(x)$ is negligibly small. Therefore, $< \Psi^-|E_k(p)|\Psi^+ > \approx < \Psi^-|\epsilon - V^{(0)}|\Psi^+ >$, and consequently

$$< \Psi^-|H-\epsilon|\Psi^+ > \approx < \Psi^-|V^{(1)}|\Psi^+ > = \frac{1}{T}\int_0^{2\pi} \frac{V^{(1)}}{\frac{d}{dp}E_k|_{p=p(x)}} \exp(2is(x)/\hbar)dx. \tag{A.2.16}$$

Observing that $V^{(1)}(x) \approx 0$ and $\bigwedge_x^j V^{(1)}(x) = \bigwedge_x^j V(x)$ for arbitrary $x \in [0, 2\pi)$ and $j \geq 0$, by partial integrating Eq. (A. 2.16) for successive $k+1$ times we obtain

$$< \Psi^-|H-\epsilon|\Psi^+ > = \frac{(i\hbar)^{k+1}}{2^{k+1}T} \sum_{j=1}^{N} \frac{\exp(2is(x_j^*)/\hbar)}{p^{k+1}\frac{d}{dp}E_k|_{p=p(x_j^*)}} \bigwedge_x^k V(x_j^*) + o(\hbar^{k+1}), \tag{A.2.17}$$

which immediately leads to Eq. (2.4).

Finally, we shall comment that although the exact eigenstates have power-law tails beyond CPR, the leading term of $\Delta\epsilon$ actually dose not relies on this detail. In fact, Eq. (A. 2.16) essentially equals to

$$\sum_{\mu\hbar, -\nu\hbar \in \mathrm{CPR}} \phi_{-\nu}^+ V_{\mu-\nu} \phi_\mu^+,$$

which is in nature controlled by the power-law decay of $\{V_m\}$ but $\{\phi_r^+\}$. Therefore, Eq. (A. 2.16) can be reproduced from the highly localized semi-classical eigenfunctions generated by smoothed Hamiltonian $H^{(0)} = E_k(p) + V^{(0)}(x)$.